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**Abstract**

An explicit asymptotic solution to the elasticity system in a three-dimensional domain in the vicinity of an elliptical crack front, or for an elliptical sharp V-notch is still unavailable. Towards its derivation we first consider the explicit asymptotic solutions of the Laplace equation in the vicinity of an elliptical singular edge in a three-dimensional domain. Both homogeneous Dirichlet and Neumann boundary conditions on the surfaces intersecting at the elliptical edge are considered. The dual singular solution is also provided to be used in a future study to extract the edges flux intensity functions by the quasi-dual-function method. We show that just as for the circular edge case, the solution in the vicinity of an elliptical edge is composed of three series, with eigenfunctions being functions of two coordinates.

*Keywords:* 3-D singularities, elliptical singular edge, edge flux intensity functions

## 1. Introduction

Asymptotic solutions of the elasticity system or the Laplace equation in the vicinity of singular points over two dimensional domains have been investigated for over half a century. For the

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Laplace equation these are described by an infinite series of the eigenpairs $(\alpha_k, \phi_k(\varphi))$ and their coefficients named flux intensity factors $A_k$ (FIFs) as follows:

$$\tau(\rho, \varphi) = \sum_k A_k \rho^{\alpha_k} \phi_k(\varphi). \tag{1}$$

In three-dimensional domains such as polyhedra, both vertex and edge singularities exist, see [3, 18]. For straight edges an explicit representation of the singular solutions along the edge was provided in [5, 2, 11, 19]. It is given by two infinite series containing the 2-D series and another series of the derivative of the FIFs $A_k$ which are functions along the straight singular edge as follows:

$$\tau(\rho, \varphi, z) = \sum_k \sum_{\ell=0,2,4,6\cdots}^{\infty} \partial_z^\ell A_k(z) \rho^{\alpha_k+\ell} \phi_{k,\ell}(\varphi). \tag{2}$$

It was shown in [20] that for three-dimensional domains with a circular singular edges, the solution of the Laplace equation is given by three series:

$$\tau(\rho, \varphi, \theta) = \sum_k \sum_{\ell=0,2,4,6\cdots}^{\infty} \partial_\theta^\ell A_k(\theta) \rho^{\alpha_k} \left(\frac{\rho}{R}\right)^\ell \sum_{i=0}^{\infty} \left(\frac{\rho}{R}\right)^i \phi_{k,\ell,i}(\varphi), \tag{3}$$

where $\theta$ is the angle surround the circular edge, $\rho$ is the normal distance from the circular edge and $R$ is the radius of the circular edge. $A_k(\theta)$ are the Edge Flux Intensity Functions (EFIFs), $\alpha_k$ are the 2-D eigenvalues and $\phi_{k,\ell,i}(\varphi)$ are the 2-D eigenfunctions and their shadow functions. This expansion contains the 3-D straight edge series and another series associated to the curvature of the singular edge.

Explicit asymptotic solutions in the vicinity of singular *elliptical* edges in a 3-D domain are not yet available neither for the elasticity system nor for the Laplace equation. The elasticity asymptotic solution has same characteristics as the Laplace asymptotic solution and towards its derivation we here consider first the Laplace equation in the vicinity of an elliptical singular edge in a three-dimensional domain. Thereafter same techniques may be used to derive the asymptotic solution for the elasticity system.

Sadowsky and Sternberg [13] considered the stress around a 3-D ellipsoidal cavity, but these ellipsoids do not contain singular edges. Green and Sneddon were the first (to the best of our knowledge) to analyze an elliptical crack in an infinite body under uniform tension [4]. They used the solutions in [13] to compute the normal displacement to the crack plane. Irwin [6] derived the pointwise edge stress intensity factor $K_I$ based on [4] for an elliptical crack in an infinite body under uniform tension. Kassir and Sih computed mode II and mode III edges stress intensity functions $K_{II}$ and $K_{III}$ for an elliptical crack in an infinite body under uniform shear (see [7]). These solutions have been used in many papers for computing the edge stress intensity functions by the superposition method ([14, 15]), the weight function method (for example see [10, 16]) and by Fourier approximation of the boundary condition or of the geometry (see [17, 12, 16, 9]). Leblond and Torlai provided the mechanism for the point-wise derivation of the elastic solution up to second order for a general curved crack ([8]). Much attention has been devoted to the computation of the first coefficient of the asymptotic series in elasticity, namely the pointwise evaluation of the mode I stress intensity function along an elliptical crack. In [1] the exact analytical pointwise stress intensity function has been obtained for an elliptical crack embedded in an infinite elastic body and subjected to an arbitrary applied normal stress (Mode I). However, the general asymptotic solution of the Laplace equation and the elasticity system in the vicinity of an elliptical singular edge is yet unknown. As a result there are no methods that may provide the flux/stress intensity factors along the edge as functions of the coordinate along it. Towards this goal we herein consider first the general expression of the asymptotic solution to the Laplace equation in a domain that contains an elliptical singular edge.

For the Laplace equation we show that as for the circular edge case, the solution in the vicinity of an elliptical edge is composed of three series, but here the eigenfunctions $\phi_{k,\ell,i}$ are functions of the coordinate $\varphi$ *and of the coordinate* $\gamma$ *along the elliptical edge*. Specific examples for an elliptical crack with homogeneous Dirichlet or Neumann BCs are provided. The asymptotic solution and especially the dual solution is required to extract the edge flux intensity function (EFIFs) using the quasi-dual function method as in [2, 20]. Therefore we explicit provide in Appendix Appendix A the first terms of the dual solution. This is the first step for the computation of the solution to the elasticity system in the vicinity of elliptical singular edges which is of major engineering importance because most surface cracks are semi-elliptical.

## 2. The Laplace equation in the vicinity of an elliptic singular edge

Consider a singular edge of an elliptic shape in the $x_1 - x_2$ plane. We use a standard orthogonal curvilinear elliptic coordinate system $(\beta, \gamma)$ related to the Cartesian coordinates by:

$$x_1 = a\cosh(\beta)\cos(\gamma) \quad (4)$$

$$x_2 = a\sinh(\beta)\sin(\gamma) \quad (5)$$

where $\beta \geq 0$, $\gamma \in [0, 2\pi]$, and $\pm a$ are the ellipse foci. A planar ellipse is determined by a fixed $\beta = \beta_0$ and a given $a$. The outer normal vector $\hat{n}$ at any point along such an ellipse (determined by $\gamma$) is:

$$\hat{n} = \frac{\sqrt{2}}{\sqrt{\cosh(2\beta_0) - \cos(2\gamma)}} \left(\sinh(\beta_0)\cos(\gamma)\hat{x}_1, \quad \cosh(\beta_0)\sin(\gamma)\hat{x}_2, \quad 0\right) \quad (6)$$

**Remark 1.** *The ellipse reduces to a circle of a radius $R$ by substituting $a = \frac{R}{\cosh(\beta_0)}$ or $a = \frac{R}{\sinh(\beta_0)}$ at the limit $\beta_0 \to \infty$ in* (4)-(5).

We locate the two orthogonal coordinates $(\rho, \varphi)$ initiating at the elliptical edge in the plane containing the normal $\hat{n}$ as shown in Fig. 1. The connection between the coordinates $\rho, \gamma, \varphi$

and the Cartesian coordinates $x_1, x_2, x_3$ is:

$$x_1 = \left(a\cosh(\beta_0) + \rho\cos(\varphi)\frac{\sqrt{2}\sinh(\beta_0)}{\sqrt{\cosh(2\beta_0) - \cos(2\gamma)}}\right)\cos(\gamma) \tag{7}$$

$$x_2 = \left(a\sinh(\beta_0) + \rho\cos(\varphi)\frac{\sqrt{2}\cosh(\beta_0)}{\sqrt{\cosh(2\beta_0) - \cos(2\gamma)}}\right)\sin(\gamma) \tag{8}$$

$$x_3 = \rho\sin(\varphi) \tag{9}$$

and the scale factors (also known as the Lamé factors) are:

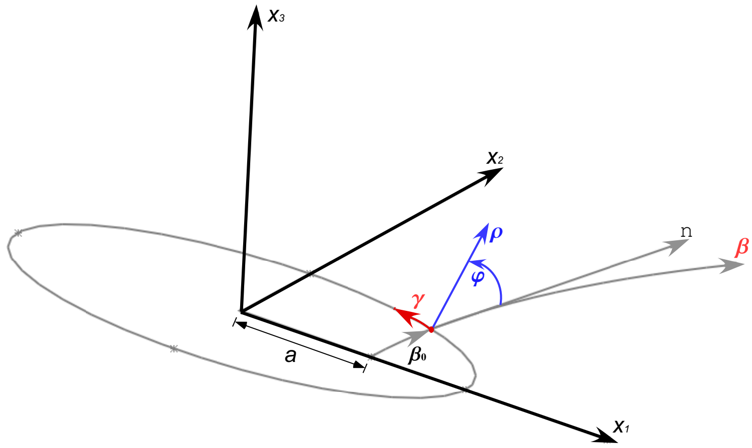


Figure 1: An ellipse in the x-y plane and the notations.

$$h_\rho = 1 \tag{10}$$

$$h_\varphi = \rho \tag{11}$$

$$h_\gamma = \frac{a}{q(\gamma)}\left(g(\gamma) + \frac{\rho}{a}f(\varphi)\right) \tag{12}$$

with:

$$f(\varphi) = \sinh(2\beta_0)\cos(\varphi) \tag{13}$$

$$q(\gamma) = \cosh(2\beta_0) - \cos(2\gamma) \tag{14}$$

$$g(\gamma) = \sqrt{q^3(\gamma)/2} \tag{15}$$

The Laplace equation in the new coordinate system $(\rho, \varphi, \gamma)$ located on the ellipse edge (see Figure 1 ) is:

$$\begin{aligned}
\Delta_{\{\rho,\varphi,\gamma\}} &= \partial_{\rho\rho} + \frac{1}{\rho}\partial_\rho + \frac{1}{\rho^2}\partial_{\varphi,\varphi} \\
&+ \frac{1}{a}\left(\frac{\sinh(2\beta_0)}{g(\gamma)+\frac{\rho}{a}f(\varphi)}\right)\left(\cos(\varphi)\partial_\rho - \frac{1}{\rho}\sin(\varphi)\partial_\varphi\right) \\
&- \frac{1}{a^2}\left(\frac{1}{g(\gamma)+\frac{\rho}{a}f(\varphi)}\right)^2 q(\gamma)\sin(2\gamma)\partial_\gamma \\
&+ \frac{\rho}{a^3}\left(\frac{1}{g(\gamma)+\frac{\rho}{a}f(\varphi)}\right)^3 q(\gamma)\sin(2\gamma)3f(\varphi)\partial_\gamma \\
&+ \frac{1}{a^2}\left(\frac{q(\gamma)}{g(\gamma)+\frac{\rho}{a}f(\varphi)}\right)^2 \partial_{\gamma\gamma}
\end{aligned} \tag{16}$$

**Remark 2.** *For the limit case of a circular crack* $\left(a = \frac{R}{\cosh(\beta_0)}\right.$ *or* $a = \frac{R}{\sinh(\beta_0)}$ *and* $\left.\beta_0 \to \infty\right)$ *:*

$$\lim_{\beta_0\to\infty} \left\{\frac{1}{a}\left(\frac{1}{g(\gamma)+\frac{\rho}{a}f(\varphi)}\right)\sinh(2\beta_0)\right\} = \frac{1}{R}\frac{1}{1+\frac{\rho}{R}\cos(\varphi)}$$

$$\lim_{\beta_0\to\infty} \left\{\frac{1}{a^2}\left(\frac{1}{g(\gamma)+\frac{\rho}{a}f(\varphi)}\right)^2 q(\gamma)\sin(2\gamma)\right\} = 0$$

$$\lim_{\beta_0\to\infty} \left\{\frac{\rho}{a^3}\left(\frac{1}{g(\gamma)+\frac{\rho}{a}f(\varphi)}\right)^3 3q(\gamma)\sin(2\gamma)f(\varphi)\right\} = 0$$

$$\lim_{\beta_0\to\infty} \left\{\frac{1}{a^2}\left(\frac{q(\gamma)}{g(\gamma)+\frac{\rho}{a}f(\varphi)}\right)^2\right\} = \frac{1}{R^2}\frac{1}{\left(1+\frac{\rho}{R}\cos(\varphi)\right)^2}$$

*and* (16) *reduces to the Laplace equation along a circular edge as presented in [20]).*

Multiplying $\Delta\tau = 0$ by $\left(1 + \frac{\rho}{a\ g(\gamma)} f(\varphi)\right)^3 \rho^2 \neq 0$, (16) becomes:

$$\begin{aligned} \left(1 + \frac{\rho f(\varphi)}{a g(\gamma)}\right)^3 \rho^2 \times \Delta \quad &= \quad M_0 + \left(\frac{\rho \sinh(2\beta_0)}{a g(\gamma)}\right)(3\cos\varphi M_0 + \rho M_{01}) \qquad (17) \\ &+ \quad \left(\frac{\rho \sinh(2\beta_0)}{a g(\gamma)}\right)^2 \left(3\cos^2\varphi M_0 + 2\cos\varphi M_{01} + \frac{1}{\sinh(2\beta_0)^2}\left(-q(\gamma)\sin(2\gamma)\partial_\gamma + q(\gamma)^2 \partial_{\gamma\gamma}\right)\right) \\ &+ \quad \left(\frac{\rho \sinh(2\beta_0)}{a g(\gamma)}\right)^3 \left(\cos^3\varphi M_0 + \cos^2\varphi M_{01} + \frac{\cos\varphi}{\sinh(2\beta_0)^2}\left(2q(\gamma)\sin(2\gamma)\partial_\gamma + q(\gamma)^2 \partial_{\gamma\gamma}\right)\right) \end{aligned}$$

with:

$$M_0 \quad = \quad (\rho\partial_\rho)^2 + \partial_{\varphi\varphi} \qquad (18)$$

$$M_1 \quad = \quad \cos(\varphi)\rho\partial_\rho - \sin(\varphi)\partial_\varphi \qquad (19)$$

Following the Laplace solution for the circular edge (see [20]), we consider a solution of (17) of the form:

$$\tau = \sum_k \sum_{\ell=0}^{\infty} \partial_\gamma^\ell A_k(\gamma) \rho^{\alpha_k} \left(\frac{\rho\ q(\gamma)}{a\ g(\gamma)}\right)^\ell \sum_{i=0}^{\infty} \left(\frac{\rho\ \sinh(2\beta_0)}{a\ g(\gamma)}\right)^i \phi_{k,\ell,i}(\varphi,\gamma) \qquad (20)$$

with either homogeneous Dirichlet or Neumann BCs on $\Gamma_1$ and $\Gamma_2$ (see Fig. 2):

**Remark 3.** *For the limit case of circular crack:*

$$\lim_{\beta_0 \to \infty} \left(\frac{\rho\ \sinh(2\beta_0)}{a\ g(\gamma)}\right)^i = \left(\frac{\rho}{R}\right)^i$$

$$\lim_{\beta_0 \to \infty} \left(\frac{\rho\ q(\gamma)}{a\ g(\gamma)}\right)^\ell = \left(\frac{\rho}{R}\right)^\ell$$

*and* (20) *reduces to* (3).

The homogeneous Dirichlet boundary conditions on $\varphi_1$ and $\varphi_2$ in view of (20) are:

$$\phi_{k,\ell,i}(\varphi_1,\gamma) = \phi_{k,\ell,i}(\varphi_2,\gamma) = 0 \ \text{ on } \Gamma_1 \cup \Gamma_2 \ \ \forall\, \gamma \in [0, 2\pi], \qquad (21)$$

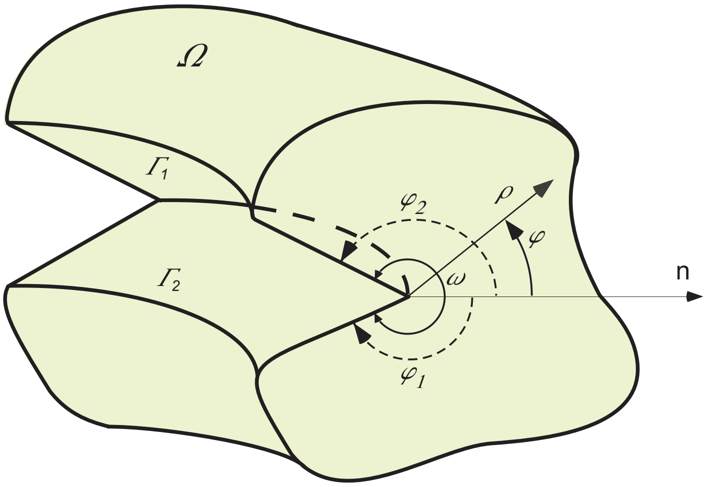


Figure 2: A slice of the ellipse singular edge.

and the homogeneous Neumann boundary conditions are:

$$\partial_\varphi \phi_{k,\ell,i}(\varphi_1, \gamma) = \partial_\varphi \phi_{k,\ell,i}(\varphi_2, \gamma) = 0 \text{ on } \Gamma_1 \cup \Gamma_2 \ \ \forall \gamma \in [0, 2\pi]. \tag{22}$$

In Figure 3 the boundaries on which homogeneous Dirichlet or Neumann boundary conditions are prescribed are visualized.

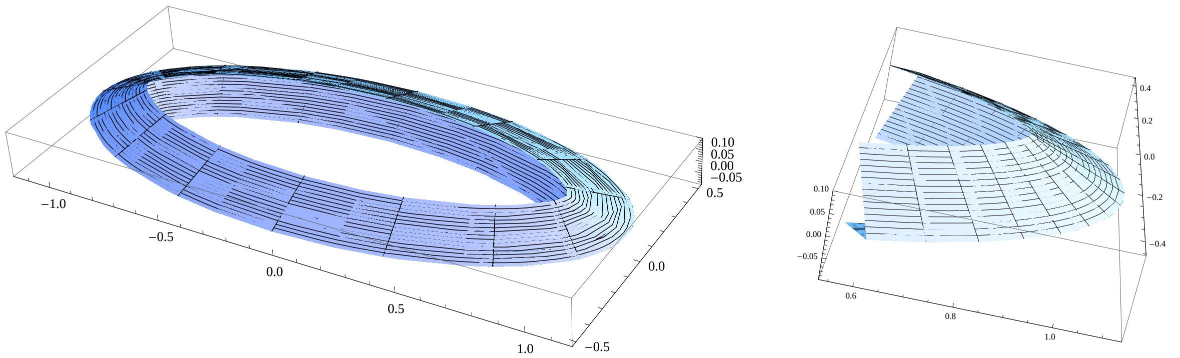


Figure 3: The surfaces that intersect at the elliptic singular edge on which homogeneous boundary conditions are prescribed. Left - The entire ellipse, Right - Zoom in the right part ( $\gamma = 0$ ).

Substituting (20) in (17) one obtains:

$$\begin{aligned}
0 \;=\;& A_k(\gamma)\rho^{\alpha_k}\Big[\{m_0\phi_{k,0,0}\} \\
&+\left(\frac{\rho\,\sinh(2\beta_0)}{a\,g(\gamma)}\right)\{m_0\phi_{k,0,1}+(3\cos(\varphi)\,m_0+m_{01})\,\phi_{k,0,0}\} \\
&+\left(\frac{\rho\,\sinh(2\beta_0)}{a\,g(\gamma)}\right)^2\Big\{m_0\phi_{k,0,2}+(3\cos(\varphi)\,m_0+m_{01})\,\phi_{k,0,1}+\big(3\cos^2(\varphi)\,m_0+2\cos(\varphi)\,m_{01}\big)\,\phi_{k,0,0} \\
&\qquad+\frac{q(\gamma)}{\sinh^2(2\beta_0)}\left(-\sin(2\gamma)\partial_\gamma+q(\gamma)\partial_{\gamma\gamma}\right)\phi_{k,0,0}\Big\} \\
&+\left(\frac{\rho\,\sinh(2\beta_0)}{a\,g(\gamma)}\right)^3\Big\{m_0\phi_{k,0,3}+(3\cos(\varphi)\,m_0+m_{01})\,\phi_{k,0,2}+\big(3\cos^2(\varphi)m_0+2\cos(\varphi)m_{01}\big)\,\phi_{k,0,1} \\
&\qquad+\big(\cos^3(\varphi)m_0+\cos^2(\varphi)\,m_{01}\big)\,\phi_{k,0,0} \\
&\qquad+\frac{q(\gamma)}{\sinh(2\beta_0)^2}\left(g(\gamma)\left(-\sin(2\gamma)\partial_\gamma+q(\gamma)\partial_{\gamma\gamma}\right)\left(\frac{\phi_{k,0,1}}{g(\gamma)}\right)+\cos(\varphi)\left(2\sin(2\gamma)\partial_\gamma+q(\gamma)\partial_{\gamma\gamma}\right)\phi_{k,0,0}\right)\Big\}+\cdots\Big]
\end{aligned} \tag{23}$$

$$\begin{aligned}
+\;& A'_k(\gamma)\rho^{\alpha_k}\left(\frac{\rho\,q(\gamma)}{a\,g(\gamma)}\right)\Big[\{m_0\phi_{k,1,0}\} \\
&+\left(\frac{\rho\,\sinh(2\beta_0)}{a\,g(\gamma)}\right)\left\{m_0\phi_{k,1,1}+(3\cos(\varphi)m_0+m_{01})\,\phi_{k,1,0}+\frac{1}{\sinh(2\beta_0)}\left(-\sin(2\gamma)+2q(\gamma)\partial_\gamma\right)\phi_{k,0,0}\right\} \\
&+\left(\frac{\rho\,\sinh(2\beta_0)}{a\,g(\gamma)}\right)^2\Big\{m_0\phi_{k,1,2}+(3\cos(\varphi)m_0+m_{01})\,\phi_{k,1,1}+\big(3\cos^2(\varphi)\,m_0+2\cos(\varphi)m_{01}\big)\,\phi_{k,1,0}+ \\
&\qquad\frac{1}{\sinh(2\beta_0)}\left(-\sin(2\gamma)\phi_{k,0,1}+2q(\gamma)g(\gamma)\partial_\gamma\left(\frac{\phi_{k,0,1}}{g(\gamma)}\right)+2\cos(\varphi)\left(\sin(2\gamma)+q(\gamma)\partial_\gamma\right)\phi_{k,0,0}\right) \\
&\qquad+\frac{g(\gamma)}{\sinh(2\beta_0)^2}\left(-\sin(2\gamma)\partial_\gamma+q(\gamma)\partial_{\gamma\gamma}\right)\left(\frac{q(\gamma)\phi_{k,1,0}}{g(\gamma)}\right)\Big\}+\cdots\Big]
\end{aligned} \tag{24}$$

$$
\begin{aligned}
+ \quad & A_k''(\gamma)\rho^{\alpha_k}\left(\frac{\rho\, q(\gamma)}{a\, g(\gamma)}\right)^2 \left[\{m_0\phi_{k,2,0}+\phi_{k,0,0}\}\right.\\
& +\left(\frac{\rho\ \sinh(2\beta_0)}{a\, g(\gamma)}\right)\{m_0\phi_{k,2,1}+(3\cos(\varphi)\, m_0+m_{01})\,\phi_{k,2,0}+\phi_{k,0,1}\\
& \qquad\qquad\left.+\cos(\varphi)\phi_{k,0,0}+\frac{1}{\sinh(2\beta_0)}\left(-\sin(2\gamma)\phi_{k,1,0}+2g(\gamma)\partial_\gamma\left(\frac{q(\gamma)\phi_{k,1,0}}{g(\gamma)}\right)\right)\right\}+\cdots\Big]\\
+ \quad & A_k'''(\gamma)\rho^{\alpha_k}\left(\frac{\rho\, q(\gamma)}{a\, g(\gamma)}\right)^3\left[\{m_0\phi_{k,3,0}+\phi_{k,1,0}\}+\cdots\right]\\
+ \quad & \cdots
\end{aligned}
$$

where $m_0$ and $m_1$ are defined by:

$$
m_0\phi_{k,l,i} \triangleq (\alpha+l+i)^2\phi_{k,l,i}+\partial_{\varphi\varphi}\phi_{k,l,i} \tag{25}
$$

$$
m_{01}\phi_{k,l,i} \triangleq (\alpha+l+i)\cos(\varphi)\phi_{k,l,i}-\sin(\varphi)\partial_\varphi\phi_{k,l,i} \tag{26}
$$

Equation (23) has to hold true for any $(\rho/a)^i$ and for any $\partial_\gamma^\ell A_k(\gamma)$, resulting in the following recursive set of ODEs for the determination of $\phi_{k,\ell,i}(\varphi,\gamma)$:

$$
m_0\phi_{k,0,0} = 0 \tag{27}
$$

$$
m_0\phi_{k,0,1} = -\left(3f(\varphi)m_0+m_1\right)\phi_{k,0,0} \tag{28}
$$

$$
\begin{aligned}
m_0\phi_{k,0,2} = & -\left(3\cos(\varphi)m_0+m_{01}\right)\phi_{k,0,1}-\cos(\varphi)\left(3\cos(\varphi)m_0+2m_{01}\right)\phi_{k,0,0}\\
& +\frac{q(\gamma)}{\sinh^2(2\beta_0)}\left(\sin(2\gamma)\partial_\gamma-q(\gamma)\partial_{\gamma\gamma}\right)\phi_{k,0,0}
\end{aligned} \tag{29}
$$

$$
\begin{aligned}
m_0\phi_{k,0,3} = & -\left(3\cos(\varphi)m_0+m_{01}\right)\phi_{k,0,2}-\cos(\varphi)\left(3\cos(\varphi)m_0+2m_{01}\right)\phi_{k,0,1}-\cos^2(\varphi)\left(\cos(\varphi)m_0+m_{01}\right)\phi_{k,0,0}\\
& -\frac{q(\gamma)}{\sinh^2(2\beta_0)}\left(g(\gamma)\left(-\sin(2\gamma)\partial_\gamma+q(\gamma)\partial_{\gamma\gamma}\right)\left(\frac{\phi_{k,0,1}}{g(\gamma)}\right)+\cos(\varphi)\left(2\sin(2\gamma)\partial_\gamma+q(\gamma)\partial_{\gamma\gamma}\right)\phi_{k,0,0}\right)
\end{aligned} \tag{30}
$$

$$
m_0\phi_{k,1,0} = 0 \tag{31}
$$

$$
m_0\phi_{k,1,1} = -\left(3\cos(\varphi)m_0+m_{01}\right)\phi_{k,1,0}-\frac{1}{\sinh(2\beta_0)}\left(-\sin(2\gamma)+2q(\gamma)\partial_\gamma\right)\phi_{k,0,0} \tag{32}
$$

$$
\begin{aligned}
m_0\phi_{k,1,2} = & -\left(3\cos(\varphi)m_0+m_{01}\right)\phi_{k,1,1}-\cos(\varphi)\left(3\cos(\varphi)m_0+2m_{01}\right)\phi_{k,1,0}\\
& -\frac{1}{\sinh(2\beta_0)}\left(-\sin(2\gamma)\phi_{k,0,1}+2q(\gamma)g(\gamma)\partial_\gamma\left(\frac{\phi_{k,0,1}}{g(\gamma)}\right)+2\cos(\varphi)\left(\sin(2\gamma)+q(\gamma)\partial_\gamma\right)\phi_{k,0,0}\right)\\
& -\frac{g(\gamma)}{\sinh^2(2\beta_0)}\left(-\sin(2\gamma)\partial_\gamma+q(\gamma)\partial_{\gamma\gamma}\right)\left(\frac{q(\gamma)\phi_{k,1,0}}{g(\gamma)}\right)
\end{aligned} \tag{33}
$$

$$
m_0\phi_{k,2,0} = -\phi_{k,0,0} \tag{34}
$$

$$
\begin{aligned}
m_0\phi_{k,2,1} = & -\left(3\cos(\varphi)m_0+m_{01}\right)\phi_{k,2,0}-\phi_{k,0,1}+\cos(\varphi)\phi_{k,0,0}\\
& -\frac{1}{\sinh(2\beta_0)}\left(-\sin(2\gamma)\phi_{k,1,0}+2g(\gamma)\partial_\gamma\left(\frac{q(\gamma)\phi_{k,1,0}}{g(\gamma)}\right)\right)
\end{aligned} \tag{35}
$$

$$
m_0\phi_{k,3,0} = -\phi_{k,1,0} \tag{36}
$$

$$
\cdots
$$

Equation (27) with the BCs (21) or (22) is the well known eigenvalues problem in a 2-D V-notched domain. For example, for a crack with homogeneous Neumann BCs the eigenvalues are: $\alpha_k = 0, 1/2, 1, 3/2, 2, \cdots$ . Notice that equations (27)-(30) depend only on $\varphi$ , so:

$$\phi_{k,0,0}(\varphi,\gamma) = \phi_{k,0,0}(\varphi) \tag{37}$$

$$\phi_{k,0,1}(\varphi,\gamma) = \phi_{k,0,1}(\varphi) \tag{38}$$

$$\phi_{k,0,2}(\varphi,\gamma) = \phi_{k,0,2}(\varphi) \tag{39}$$

For simplicity, we can choose the homogeneous solution of (31) and (36) as zero, so:

$$\phi_{k,1,0}(\varphi,\gamma) = 0 \tag{40}$$

$$\phi_{k,3,0}(\varphi,\gamma) = 0 \tag{41}$$

and (27)-(36) reduce to:

$$m_0\phi_{k,0,0} = 0 \tag{42}$$

$$m_0\phi_{k,0,1} = -\left(3\cos(\varphi)m_0 + m_{01}\right)\phi_{k,0,0} \tag{43}$$

$$m_0\phi_{k,0,2} = -\left(3\cos(\varphi)m_0 + m_{01}\right)\phi_{k,0,1} - \cos(\varphi)\left(3\cos(\varphi)\,m_0 + 2m_{01}\right)\phi_{k,0,0} \tag{44}$$

$$\begin{aligned} m_0\phi_{k,0,3} = {} & -\left(3\cos(\varphi)m_0 + m_{01}\right)\phi_{k,0,2} - \cos(\varphi)\left(3\cos(\varphi)\,m_0 + 2m_{01}\right)\phi_{k,0,1} \\ & -\cos^2(\varphi)\left(\cos(\varphi)m_0 + m_{01}\right)\phi_{k,0,0} + \frac{q(\gamma)}{\sinh^2(2\beta_0)} g(\gamma)\left(\sin(2\gamma)\partial_\gamma - q(\gamma)\partial_{\gamma\gamma}\right)\left(\frac{\phi_{k,0,1}}{g(\gamma)}\right) \end{aligned} \tag{45}$$

$$m_0\phi_{k,1,1} = \frac{\sin(2\gamma)}{\sinh(2\beta_0)}\phi_{k,0,0} \tag{46}$$

$$\begin{aligned} m_0\phi_{k,1,2} = {} & -\left(3\cos(\varphi)m_0 + m_{01}\right)\phi_{k,1,1} \\ & + \frac{1}{\sinh(2\beta_0)}\left(\sin(2\gamma)\left(\phi_{k,0,1} - 2\cos(\varphi)\phi_{k,0,0}\right) - 2q(\gamma)g(\gamma)\partial_\gamma\left(\frac{\phi_{k,0,1}}{g(\gamma)}\right)\right) \end{aligned} \tag{47}$$

$$m_0\phi_{k,2,0} = -\phi_{k,0,0} \tag{48}$$

$$m_0\phi_{k,2,1} = -\left(3\cos(\varphi)\,m_0 + m_{01}\right)\phi_{k,2,0} - \phi_{k,0,1} + \cos(\varphi)\phi_{k,0,0} \tag{49}$$

### *2.1. A specific example: An elliptical Crack with homogeneous Dirichlet BCs*

For elliptical cracks with homogeneous Dirichlet BCs the eigenvalues are $\alpha_k = 1/2, 1, 3/2, 2, \cdots$ . Equations (42)-(48) can be solved with the homogeneous Dirichlet BCs (21) for the different $\alpha_k$

to find $\phi_{k,\ell,i}(\varphi,\gamma)$. For example, the singular part of the solution (i.e. for $\alpha_1 = 1/2$) up to order of $(\rho)^3$ is:

$$\begin{aligned}
\tau \;=\; & A_1(\gamma)\rho^{1/2}\left\{\cos\left(\frac{\varphi}{2}\right) - \left(\frac{\rho\,\sinh(2\beta_0)}{a\,g(\gamma)}\right)\frac{1}{4}\cos\frac{\varphi}{2} + \left(\frac{\rho\,\sinh(2\beta_0)}{a\,g(\gamma)}\right)^2\left(\frac{1}{12}\cos\frac{\varphi}{2} + \frac{3}{32}\cos\frac{3\varphi}{2}\right)\right.\\
& \left. - \left(\frac{\rho\,\sinh(2\beta_0)}{a\,g(\gamma)}\right)^3\left(\frac{1}{16}\cos\frac{\varphi}{2} + \frac{1}{30}\cos\frac{3\varphi}{2} + \frac{5}{128}\cos\frac{5\varphi}{2} + \frac{1}{8}\frac{-2+\cos(4\gamma)+\cos(2\gamma)\cosh(2\beta_0)}{\sinh(2\beta_0)^2}\cos\frac{\varphi}{2}\right) + \cdots\right\}\\
& + A_1'(\gamma)\rho^{1/2}\left(\frac{\rho\,q(\gamma)}{a\,g(\gamma)}\right)\frac{\sin(2\gamma)}{\sinh(2\beta_0)}\left\{\left(\frac{\rho\,\sinh(2\beta_0)}{a\,g(\gamma)}\right)\frac{1}{6}\cos\frac{\varphi}{2} - \left(\frac{\rho\,\sinh(2\beta_0)}{a\,g(\gamma)}\right)^2\left(\frac{3}{8}\cos\frac{\varphi}{2} + \frac{4}{15}\cos\frac{3\varphi}{2}\right) + \cdots\right\}\\
& A_1''(\gamma)\rho^{1/2}\left(\frac{\rho\,q(\gamma)}{a\,g(\gamma)}\right)^2\left\{-\frac{1}{6}\cos\frac{\varphi}{2} + \left(\frac{\rho\,\sinh(2\beta_0)}{a\,g(\gamma)}\right)\left(\frac{1}{8}\cos\frac{\varphi}{2} + \frac{7}{60}\cos\frac{3\varphi}{2}\right) + \cdots\right\}\\
& + \mathcal{O}\left\{\rho^4\right\}
\end{aligned} \tag{50}$$

*2.2. A specific example: An elliptical Crack with homogeneous Neumann BCs*

For elliptical cracks with homogeneous Neumann BCs the eigenvalues are $\alpha_k = 0, 1/2, 1, 3/2, 2, \cdots$. Equations (42)-(48) can be solved with homogeneous Neumann BCs (22) for the different $\alpha_k$ to find $\phi_{k,\ell,i}(\varphi,\gamma)$. For example, the singular part of the solution (i.e. for $\alpha_1 = 1/2$) up to order of $(\rho)^3$ is:

$$\begin{aligned}
\tau \;=\; & A_1(\gamma)\rho^{1/2}\left\{\sin\frac{\varphi}{2} + \left(\frac{\rho\,\sinh(2\beta_0)}{a\,g(\gamma)}\right)\frac{1}{4}\sin\frac{\varphi}{2} + \left(\frac{\rho\,\sinh(2\beta_0)}{a\,g(\gamma)}\right)^2\left(\frac{1}{12}\sin\frac{\varphi}{2} - \frac{3}{32}\sin\frac{3\varphi}{2}\right)\right.\\
& \left. + \left(\frac{\rho\,\sinh(2\beta_0)}{a\,g(\gamma)}\right)^3\left(\frac{1}{16}\sin\frac{\varphi}{2} - \frac{1}{30}\sin\frac{3\varphi}{2} + \frac{5}{128}\sin\frac{5\varphi}{2} + \frac{1}{8}\frac{-2+\cos(4\gamma)+\cos(2\gamma)\cosh(2\beta_0)}{\sinh(2\beta_0)^2}\sin\frac{\varphi}{2}\right) + \cdots\right\}\\
& + A_1'(\gamma)\rho^{1/2}\left(\frac{\rho\,q(\gamma)}{a\,g(\gamma)}\right)\frac{\sin(2\gamma)}{\sinh(2\beta_0)}\left\{\left(\frac{\rho\,\sinh(2\beta_0)}{a\,g(\gamma)}\right)\frac{1}{6}\sin\frac{\varphi}{2} + \left(\frac{\rho\,\sinh(2\beta_0)}{a\,g(\gamma)}\right)^2\left(\frac{3}{8}\sin\frac{\varphi}{2} - \frac{4}{15}\sin\frac{3\varphi}{2}\right) + \cdots\right\}\\
& + A_1''(\gamma)\rho^{1/2}\left(\frac{\rho\,q(\gamma)}{a\,g(\gamma)}\right)^2\left\{-\frac{1}{6}\sin\frac{\varphi}{2} - \left(\frac{\rho\,\sinh(2\beta_0)}{a\,g(\gamma)}\right)\left(\frac{1}{8}\sin\frac{\varphi}{2} - \frac{7}{60}\sin\frac{3\varphi}{2}\right) + \cdots\right\}\\
& + \mathcal{O}\left\{\rho^4\right\}
\end{aligned} \tag{51}$$

## 3. Summary and Conclusions

We provided the machinery to develop an explicit asymptotic solution for the Laplace equation in the vicinity of an elliptical crack or sharp V-notch in a three dimensional domain. Specifically, for an elliptical crack with homogeneous Dirichlet or Neumann boundary conditions we provided the asymptotic series up to an order of $\rho^3$. We have also shown that at the limit when the ellipse turns into a circle the asymptotic solution simplifies to the one obtained for a penny shaped crack in [20]. The asymptotic dual solution (associated with the negative eigenvalue $-1/2$) has been also derived and will serve for the extraction of the flux intensity functions using the quasi dual function method.

The machinery and solution for the Laplace equation serves as a cornerstone for the asymptotic solution for the elasticity system, and attempts are underway to obtain an asymptotic solution to the elasticity system in the vicinity of an elliptical and semi-elliptic crack.

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## Appendix A. The dual solution for an elliptical crack

Here we provide the first dual solution (associated with the negative eigenvalues $\alpha_k < 0$) for an elliptical crack with homogenoues Dirichlet or Neumann BCs on the crack faces.

### *Appendix A.1. An elliptical crack with homogenous Dirichlet BCs*

For an elliptical crack with homogenous Dirichlet BCs the first dual eigenvalue is: $\alpha = -1/2$. Substituting $\alpha = -1/2$ in equations (42)-(48) with homogeneous Dirichlet BCs (21) one obtains the first terms of the dual solution $K^{(\alpha_1=-\frac{1}{2})}$:

$$\begin{aligned} K^{(\alpha_1=-\frac{1}{2})} \quad = \quad & B_1(\gamma)\rho^{-1/2}\left\{\cos\frac{\varphi}{2} - \left(\frac{\rho\ \sinh(2\beta_0)}{a\ g(\gamma)}\right)\frac{1}{4}\cos\frac{3\varphi}{2} + \left(\frac{\rho\ \sinh(2\beta_0)}{a\ g(\gamma)}\right)^2\frac{3}{32}\cos\frac{5\varphi}{2}\right. \\ & \left. - \left(\frac{\rho\ \sinh(2\beta_0)}{a\ g(\gamma)}\right)^3\left(\frac{5}{128}\cos\frac{7\varphi}{2} + \frac{3}{8}\frac{-2+\cos(4\gamma)+\cos(2\gamma)\cosh(2\beta_0)}{\sinh(2\beta_0)^2}\cos\frac{3\varphi}{2}\right) + \cdots\right\} \\ & + B_1'(\gamma)\rho^{-1/2}\left(\frac{\rho\, q(\gamma)}{a\ g(\gamma)}\right)\frac{\sin(2\gamma)}{\sinh(2\beta_0)}\left\{\left(\frac{\rho\ \sinh(2\beta_0)}{a\ g(\gamma)}\right)\frac{1}{2}\cos\frac{\varphi}{2} - \left(\frac{\rho\ \sinh(2\beta_0)}{a\ g(\gamma)}\right)^2\left(\frac{1}{2}\cos\frac{\varphi}{2} + \frac{9}{8}\cos\frac{3\varphi}{2}\right) + \cdots\right\} \\ & + B_1''(\gamma)\rho^{-1/2}\left(\frac{\rho\, q(\gamma)}{a\ g(\gamma)}\right)^2\left\{-\frac{1}{2}\cos\frac{\varphi}{2} + \left(\frac{\rho\ \sinh(2\beta_0)}{a\ g(\gamma)}\right)\left(\frac{1}{4}\cos\frac{\varphi}{2} + \frac{3}{8}\cos\frac{3\varphi}{2}\right) + \cdots\right\} \\ & + \mathcal{O}\left\{\rho^4\right\} \end{aligned} \tag{A.1}$$

*Appendix A.2. An elliptical crack with homogeneous Neumann BCs*

For an elliptical crack with homogenous Neumann BCs the first dual eigenvalue is: $\alpha = -1/2$. Substituting $\alpha = -1/2$ in equations (42)-(48) with the homogeneous BCs (22) one obtains the first terms of the dual solution $K^{(\alpha_1=-\frac{1}{2})}$ :

$$\begin{aligned} K^{(\alpha_1=-\frac{1}{2})} &= B_1(\gamma)\rho^{-1/2}\left\{\sin\frac{\varphi}{2} - \left(\frac{\rho\,\sinh(2\beta_0)}{a\,g(\gamma)}\right)\frac{1}{4}\sin\frac{3\varphi}{2} + \left(\frac{\rho\,\sinh(2\beta_0)}{a\,g(\gamma)}\right)^2\frac{3}{32}\sin\frac{5\varphi}{2}\right. \\ &\quad \left. - \left(\frac{\rho\,\sinh(2\beta_0)}{a\,g(\gamma)}\right)^3\left(\frac{5}{128}\sin\frac{7\varphi}{2} + \frac{3}{8}\frac{-2+\cos(4\gamma)+\cos(2\gamma)\cosh(2\beta_0)}{\sinh(2\beta_0)^2}\sin\frac{3\varphi}{2}\right) + \cdots\right\} \\ &\quad + B_1'(\gamma)\rho^{-1/2}\left(\frac{\rho\,q(\gamma)}{a\,g(\gamma)}\right)\frac{\sin(2\gamma)}{\sinh(2\beta_0)}\left\{\left(\frac{\rho\,\sinh(2\beta_0)}{a\,g(\gamma)}\right)\frac{1}{2}\sin\frac{\varphi}{2} + \left(\frac{\rho\,\sinh(2\beta_0)}{a\,g(\gamma)}\right)^2\left(\frac{1}{2}\sin\frac{\varphi}{2} - \frac{9}{8}\sin\frac{3\varphi}{2}\right) + \cdots\right\} \\ &\quad + B_1''(\gamma)\rho^{-1/2}\left(\frac{\rho\,q(\gamma)}{a\,g(\gamma)}\right)^2\left\{-\frac{1}{2}\sin\frac{\varphi}{2} - \left(\frac{\rho\,\sinh(2\beta_0)}{a\,g(\gamma)}\right)\left(\frac{1}{4}\sin\frac{\varphi}{2} - \frac{3}{8}\sin\frac{3\varphi}{2}\right) + \cdots\right\} \\ &\quad + \mathcal{O}\left\{\rho^4\right\} \end{aligned} \tag{A.2}$$

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